Basics of Geometry

Euclid's Elements 300 B.C.

• Point A | Line A B $\overleftrightarrow{AB}$ | Plane m | 3 undefined terms

Segment $\overline{AB}$ (A B) | Ray $\overrightarrow{AB}$ (A B) | Angle $\angle ABC$, $\angle CBA$, $\angle B$

Colinear - on the same line

Coplanar - on the same plane

bisector - divides into 2 equal parts

Distance between $A(x_1,y_1)$ and $B(x_2,y_2)$

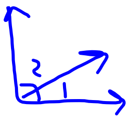
$d(AB) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Partition a Segment $(x_1+k(x_2-x_1),\ y_1+k(y_2-y_1))$

Midpoint with endpoints at (x_1,y_1) and (x_2,y_2)

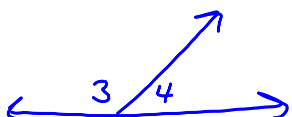
$mp = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Think average!

Complementary Angles (adjacent or not)



$m\angle 1 + m\angle 2 = 90°$

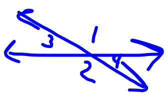
Supplementary Angles (adjacent or not)



$m\angle 3 + m\angle 4 = 180°$

Linear pairs are supplementary. (must be adjacent)

Vertical angles are equal.



$m\angle 1 = m\angle 2$

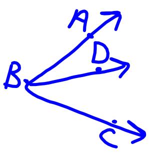
$m\angle 3 = m\angle 4$

Segment Addition Postulate

A B C

$AB + BC = AC$

Angle Addition Postulate



$m\angle ABD + m\angle DBC = m\angle ABC$

Polygon Interior Angles Theorem: sum $= (n-2)180°$, one $= \frac{(n-2)180°}{n}$ must be regular

Exterior Angles: sum $= 360°$, one $= \frac{360°}{n}$

Reasoning and Proofs

Counterexample - shows a conjecture to be False

Conditional Statement - If hypothesis, then conclusion $p \to q$

Converse $q \to p$

Biconditional $p \leftrightarrow q$
iff

Reflexive Property $a = a$

Symmetric Property If $a = b$, then $b = a$

Transitive Property If $a = b$ and $b = c$, then $a = c$

Two-column Proofs

Statements	Reasons

Paragraph Proof

Flow Proof $\boxed{\text{Statement}}$
Reason

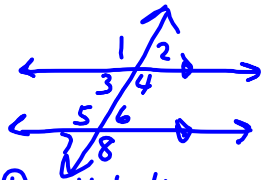
Geometric Proof

Postulate - common notion

Axiom - accepted idea

Theorem - usually needs proof

Parallel and Perpendicular Lines



Parallel lines and a transversal

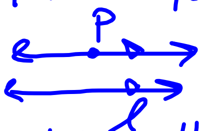
Corresponding eg. $\angle 5$ and $\angle 1$
Alternate Interior eg. $\angle 3$ and $\angle 6$
Alternate Exterior eg. $\angle 7$ and $\angle 2$
} congruent

Consecutive Interior eg. $\angle 5$ and $\angle 3$ are supplementary (same-side-interior)

Parallel Lines - Same plane, no intersection

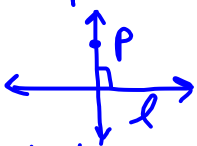
Skew Lines - different planes, no intersection

Parallel Postulate



Only 1 line through P $\parallel$ to ℓ

Perpendicular Postulate



Only 1 line through P $\perp$ to ℓ

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \text{rate of change}$$

Slope of a line through (x_1, y_1) and (x_2, y_2), $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Slope-intercept Form $y = mx + b$, m = slope, b = y-intercept

Point-slope Form $y - y_1 = m(x - x_1)$

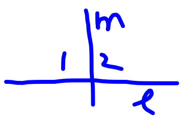
Standard Form $Ax + By = C$, $m = -\frac{A}{B}$

$\ell_1 \parallel \ell_2$ iff $m_1 = m_2$

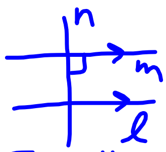
$\ell_1 \perp \ell_2$ iff $m_1 = -\frac{1}{m_2}$

eg. $m_1 = \frac{2}{5}$
$m_2 = -\frac{5}{2}$

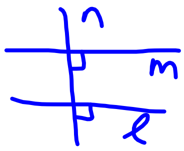
Perpendicular Theorems



If $\angle 1 \cong \angle 2$, then $\ell \perp m$

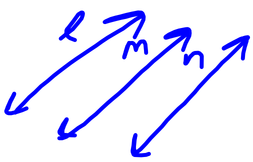


If $\ell \parallel m$ and $n \perp m$, then $n \perp \ell$



If $m \perp n$ and $\ell \perp n$, then $m \parallel \ell$

Transitive Property For Parallel Lines



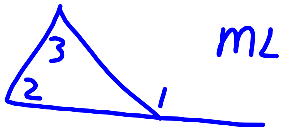
If $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$

Triangles and Congruence

Classify △'s by sides
- 3 ≅ Equilateral Triangle
- 2 or 3 ≅ Isosceles Triangle
- none ≅ Scalene Triangle

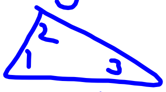
Classify △'s by angles
- Acute - 3 acute ∠'s
- Obtuse - 1 obtuse ∠
- Right - 1 right ∠

Exterior Angle Theorem

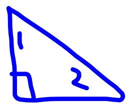


$m\angle 1 = m\angle 2 + m\angle 3$

Triangle Sum Theorem

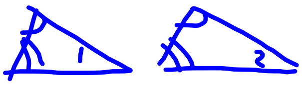


$m\angle 1 + m\angle 2 + m\angle 3 = 180°$



Complementary Sum Theorem
$m\angle 1 + m\angle 2 = 90°$

Third Angle Theorem

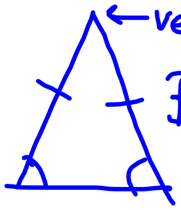


$\angle 1 \cong \angle 2$

Properties of Congruent △'s
Reflexive, Symmetric, Transitive

Prove triangles congruent by SSS, SAS, HL, ASA, and AAS.

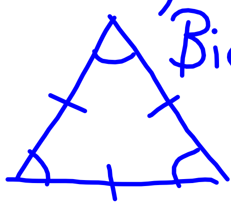
Base Angles Theorem



←vertex angle

Biconditional

Corallary



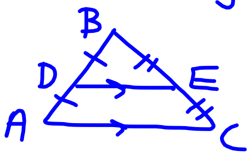
Biconditional

CPCTC - Corresponding parts of congruent triangles are congruent
(You must first prove the triangles are congruent.)

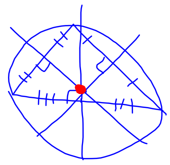
Relationships within Triangles

Midsegment Theorem

$DE \parallel AC$ and $DE = \frac{1}{2}AC$

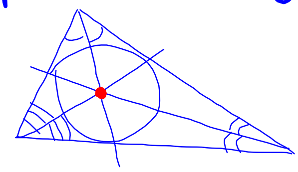


Circumcenter - intersection of perpendicular bisectors, equidistant to all 3 vertices

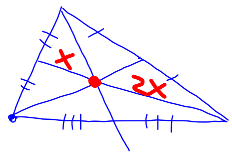


- outside if obtuse
- inside if acute
- on the midpoint of the hypotenuse if right

Incenter - intersection of angle bisectors, equidistant to all 3 sides



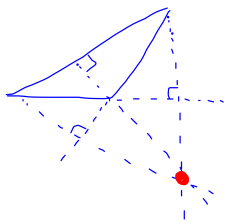
Centroid - intersection of the medians (vertex to midpoints) balance point, $\frac{2}{3}$ of vertex to mp



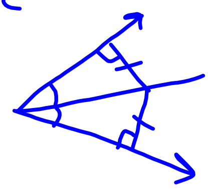
$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = \text{centroid}$$

Orthocenter - intersection of altitudes

- outside if obtuse
- inside if acute
- at right angle



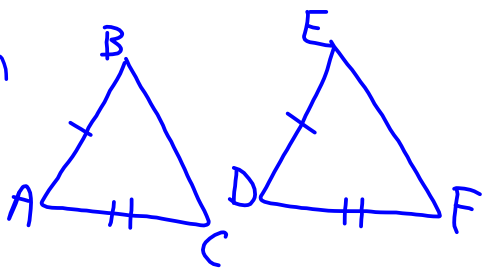
Angle Bisector Theorem and the Converse



Third Side Theorem - greater than the difference, less than the sum of the other two sides

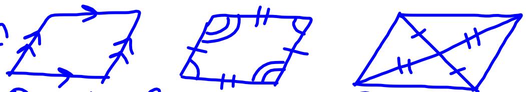
Hinge Theorem

$m\angle A > m\angle D$ iff $BC > EF$



Quadrilaterals

Parallelogram



Parallel opp. sides (definition) — Congruent Opp. Sides and angles — Diagonals bisect

Consecutive interior angles are supplementary.

Minimum conditions for a parallelogram
One pair of opposite sides parallel and congruent
Both opposite sides congruent or both opposite angles congruent.
One angle supplementary to both consecutive angles
Diagonals bisect each other.

Rectangle iff consecutive sides are perpendicular.
The diagonals are congruent.

Minimum conditions for rectangles ▱ with one right ∠
▱ with ≅ diagonals

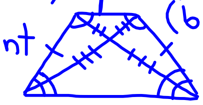
Square iff rectangle with 4 congruent sides (rhombus)

Rhombus iff parallelogram with 4 congruent sides
Diagonals are perpendicular and bisect angles

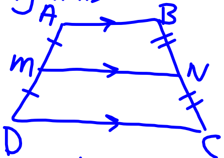
Minimum Conditions for a rhombus ▱ with ⊥ diagonals
▱ with one pair of consecutive congruent sides
▱ with diagonal bisecting opposite angles

Trapezoid-quadrilateral with only 2 parallel sides (bases)

Isosceles Trapezoid-congruent legs, base angles, and diagonals

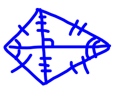


Midsegment of any trapezoid



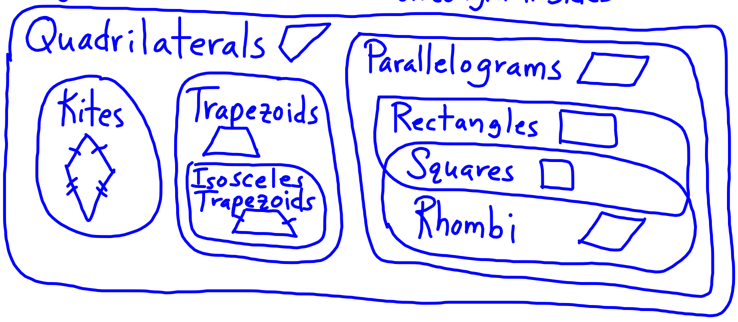
$MN = \frac{AB+CD}{2}$ average of the bases

Kite



⊥ diagonals, 2 pairs of congruent consecutive sides

Congruent angles between non congruent sides



Similarity

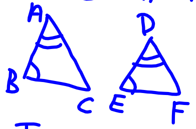
Proportion $\frac{a}{b} = \frac{c}{d}$, $ad = bc$ cross products are equal

Similar Polygons - congruent corresponding angles, proportional sides, perimeters also proportional

Similarity preserves - angle measure, betweenness, and colinearity (not distance)

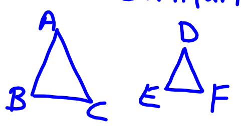
Prove triangles similar by AA, SSS, SAS (the sides must be proportional)

AA Similarity



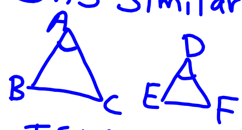
If $\angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\triangle ABC \sim \triangle DEF$

SSS Similarity



If $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\triangle ABC \sim \triangle DEF$

SAS Similarity



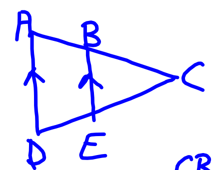
If $\angle A \cong \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$, then $\triangle ABC \sim \triangle DEF$

Similarity and congruence are NOT the same. Congruent figures are similar, but similar figures may not be congruent.

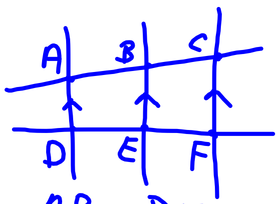
Similar - same shape

Congruent - same shape and size

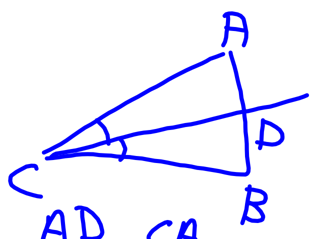
Proportionality Theorems



$AD \parallel BE$ iff $\frac{CB}{BA} = \frac{CE}{ED}$



$\frac{AB}{BC} = \frac{DE}{EF}$



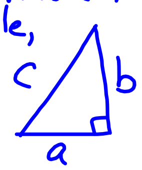
$\frac{AD}{DB} = \frac{CA}{CB}$

Right Triangles and Trigonometry

Pythagorean Theorem

If a right triangle, then $a^2+b^2=c^2$

$leg^2+leg^2=hyp^2$



Common Pythagorean Triples

3,4,5 8,15,17

5,12,13 7,24,25

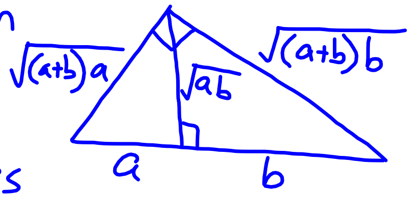
Converse of Pythagorean Theorem

If $a^2+b^2=c^2$, then right triangle

$c^2 < a^2+b^2$ acute $\triangle$

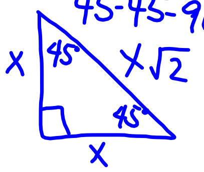
$c^2 > a^2+b^2$ obtuse $\triangle$

Geometric Mean and Altitude

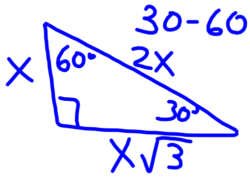


Special Triangles

45-45-90



30-60-90



hyp = 2·short leg

long leg = $\sqrt{3}$·short leg

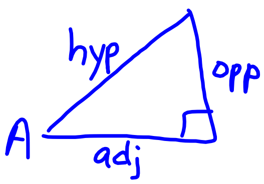
Trigonometric Ratios - Use angle to find ratio

Sine

$\sin A = \frac{opp}{hyp}$

Cosine

$\cos A = \frac{adj}{hyp}$



Tangent

$\tan A = \frac{opp}{adj}$

Inverse Trigonometric Functions - use ratio to find the angle

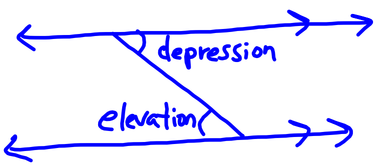
$\sin^{-1}\frac{opp}{hyp} = m\angle A$, $\cos^{-1}\frac{adj}{hyp} = m\angle A$, $\tan^{-1}\frac{opp}{adj} = m\angle A$

Your calculator will usually be in the degree mode.

The cofunctions of complementary angles are equal.

e.g. $\sin 10° = \cos 80°$

Angles of elevation and depression are measured from horizontal.



Properties of Transformations

Preimage-original figure, Image-new figure
Isometry is a congruence transformation.
(translation, reflection, and rotation)

Isometry, Rigid Motion,
Congruence Transformations
(change position, not shape or size)

Preserve distance, angle measure, betweenness, and collinearity

Some transformations change the size of the object.
Dilation- stretch or shrink to create a similar form
Dilations preserve angle measure, betweenness, and collinearity.
Scale Factor - ratio of image side to the corresponding original side

Vector = terminal point - initial point

$(2,1) \to (4,5)$ $\quad (4,5)-(2,1) = \langle 2,4 \rangle$

A tesselation is a tiling that covers the surface with no gaps or overlaps.
Only 3 regular polygons tesselate:
squares, triangles, and hexagons

Measuring Length and Area

Area Square $A = s^2$ Triangle $A = \frac{1}{2}bh$

Rectangle $A = bh$ Trapezoid $A = \frac{1}{2}h(b_1 + b_2)$

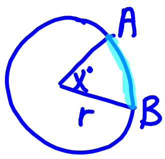
Parallelogram $A = bh$ Kite or Rhombus $A = \frac{1}{2}d_1 d_2$

Ratios of Similar Polygons

$$\frac{\text{Side 1}}{\text{Side 2}} = \frac{a}{b} \qquad \frac{\text{Area 1}}{\text{Area 2}} = \frac{a^2}{b^2} \text{ or } \left(\frac{a}{b}\right)^2$$

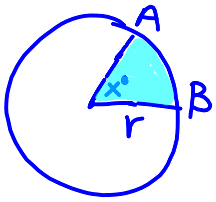
Circumference $C = \pi d = 2\pi r$ Area $A = \pi r^2$

Arc Length =



$$\frac{x°}{360°} \cdot 2\pi r$$

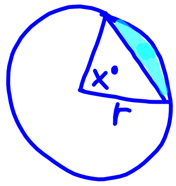
Sector Area =



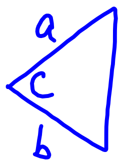
$$\frac{x°}{360°} \cdot \pi r^2$$

Segment Area = Sector Area − Triangle Area

$$\frac{x°}{360°} \cdot \pi r^2 - \frac{1}{2} r^2 \sin x°$$



Area of a Triangle

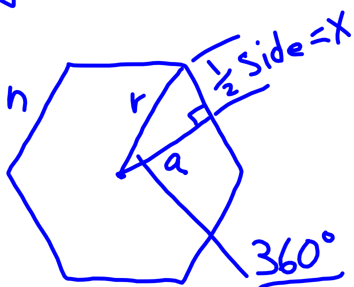


$A = \frac{1}{2}ab\sin C$

Area of a Regular Polygon

$A = \frac{1}{2}aP$ a = apothem

P = perimeter



$\frac{360°}{2n}$ or $\frac{180°}{n}$

Geometric Probability

$$= \frac{\text{desired length}}{\text{total length}} \text{ or } \frac{\text{desired area}}{\text{total area}}$$

Surface Area and Volume of Solids

Polyhedron - a solid with polygon faces

Platonic Solids - use the same regular polygon for all faces

Regular Tetrahedron	Cube	Regular Octahedron	Regular Dodecahedron	Regular Icosahedron
4-Triangles	6-Squares	8-Triangles	12-Pentagons	20-Triangles

Euler's Theorem $F+V=E+2$

Prism $SA=2B+Ph$ (for a right prism)

$=2(\text{Base area})+(\text{Base perimeter})\text{height}$

$V=Bh=(\text{Base area})\text{height}$

Pyramid $SA=B+\frac{1}{2}P\ell$ (for a right pyramid)

$=(\text{Base area})+\frac{1}{2}(\text{Base perimeter})(\text{lateral height})$

$V=\frac{1}{3}Bh=\frac{1}{3}(\text{Base area})\text{height}$

Cylinder $SA=2\pi r^2+2\pi rh$ (for a right cylinder)

$V=\pi r^2 h$

Cone $SA=\pi r^2+\pi r\ell$, $V=\frac{1}{3}\pi r^2 h$
(for a right cone)

Sphere $SA=4\pi r^2$, $V=\frac{4}{3}\pi r^3$

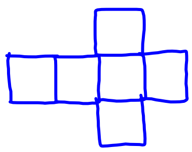
Break down composite solids.

Ratios for Similar Solids

Linear	Surface Area	Volume
$\frac{a}{b}$	$\left(\frac{a}{b}\right)^2$	$\left(\frac{a}{b}\right)^3$

Cavalieri's Principle - if the same cross-section and height, then the same volume

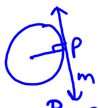
Net - a 2 dimensional shape that can be folded to form a three dimensional shape



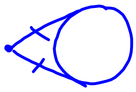
Density - the ratio of mass to volume $d=\frac{m}{V}$

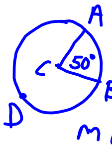
Properties of Circles

Tangent Line m is $\perp$ to the radius at P



Tangents from a common point

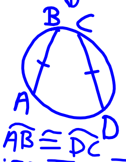




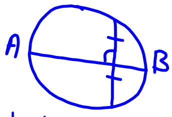
minor arc $m\overset{\frown}{AB} = 50°$

major arc $m\overset{\frown}{ADB} = 310°$

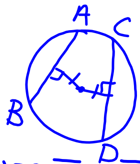
$m\overset{\frown}{AB} + m\overset{\frown}{ADB} = 360°$



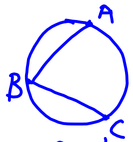
$\overset{\frown}{AB} \cong \overset{\frown}{DC}$ iff $\overline{AB} \cong \overline{DC}$



$\perp$ chord bisected iff $\overline{AB}$ diameter

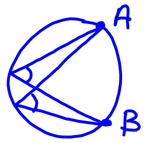


iff $\overline{AB} \cong \overline{DC}$

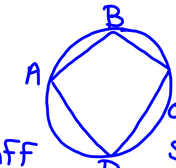


$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$

Inscribed Angle

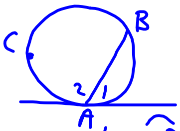


Same arc iff congruent $\angle$'s



Inscribed quadrilateral opposite $\angle$'s supplementary

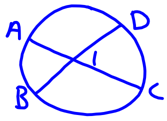
$m\angle A + m\angle C = m\angle B + m\angle D = 180°$



$m\angle 1 = \frac{1}{2} m\overset{\frown}{AB}$

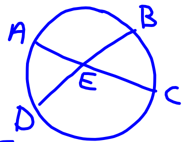
$m\angle 2 = \frac{1}{2} m\overset{\frown}{ACB}$

Angles Inside



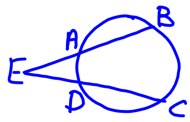
$m\angle 1 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$

Segments of Chords

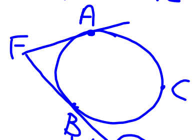


$AE \cdot EC = DE \cdot EB$

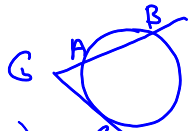
Angles Outside the Circle



$m\angle E = \frac{1}{2}(m\overset{\frown}{BC} - m\overset{\frown}{AD})$



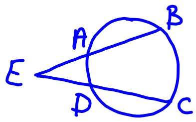
$m\angle F = \frac{1}{2}(m\overset{\frown}{ACB} - m\overset{\frown}{AB})$



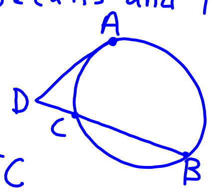
$m\angle G = \frac{1}{2}(m\overset{\frown}{BC} - m\overset{\frown}{AC})$

$\frac{1}{2}$(Furthest arc − closest arc)

Segments of Secants and Tangents



$EA \cdot EB = ED \cdot EC$



$DA^2 = DC \cdot DB$

outside * whole = outside * whole

Standard Form Equation For a Circle

$(x-h)^2 + (y-k)^2 = r^2$, Center $= (h,k)$, Radius $= r$

General Form Cartesian Equation

$Ax^2 + Cy^2 + Dx + Ey + F = 0$ Complete the square to find the center and radius.

1 revolution $= 360° = 2\pi$ radians

Rad → Deg use $\frac{180°}{\pi}$

Deg → Rad use $\frac{\pi}{180°}$

$S = r\theta$

S: arc length; r: radius; θ: central angle in radians